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Global Existence and Asymptotic Behavior of Solutions of Second-Order Nonlinear Differential Equations

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1. INTRODUCTION

In this paper we consider the second-order nonlinear differential equation

$$(p(t)\, y')' = f(t, y, y'), \qquad t \geqslant 0, \tag{1}$$

under the following standing assumptions on p and f:

(A1) p: $[0, \infty) \to (0, \infty)$ is continuous and

$$P(t) = \int_0^t \frac{ds}{p(s)} \to \infty \qquad \text{as} \quad t \to \infty;$$

(A2) f: $[0, \infty) \times R \times R \to (0, \infty)$ is continuous and $f(t, u, v)$ is nondecreasing in u and v;

(A3) Every Cauchy problem for (1) has a unique solution.

A prototype of Eq. (1) satisfying (A1)–(A3) is

$$y'' = a(t)\, e^{y}, \tag{2}$$

or more generally

$$y'' = a(t) \exp\left(\sum_{i=1}^{m} b_i(t)\, y^{2i-1} + \sum_{j=1}^{n} c_j(t)(y')^{2j-1} \right), \tag{3}$$

where a, b_i, and c_j are positive continuous functions on $[0, \infty)$.

Let $y(t)$ be a solution of (1) with the maximal interval of existence $[0, T_y)$. From (1) we have $(p(t)\, y'(t))' > 0$, so that $p(t)\, y'(t)$ is increasing on $[0, T_y)$. It happens that either $T_y < \infty$ and $\lim_{t \to T_y-} p(t)\, y'(t) = \infty$, or

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else $T_y = \infty$ and $\lim_{t\to\infty} p(t)\,y'(t)$ exists in $R \cup \{\infty\}$. In the former case $y(t)$ is called a *singular solution*, and in the latter $y(t)$ is called a *proper solution*. The set of proper solutions of (1) is further classified into the following four classes:

(i) the class of *strongly increasing solutions* consisting of all solutions $y(t)$ such that $\lim_{t\to\infty} p(t)\,y'(t) = \infty$;

(ii) the class of *weakly increasing solutions* consisting of all solutions $y(t)$ such that $\lim_{t\to\infty} p(t)\,y'(t) \in (0, \infty)$.

(iii) the class of *weakly decreasing solutions* consisting of all solutions $y(t)$ such that $\lim_{t\to\infty} p(t)\,y'(t) = 0$;

(iv) the class of *strongly decreasing solutions* consisting of all solutions $y(t)$ such that $\lim_{t\to\infty} p(t)\,y'(t) \in (-\infty, 0)$.

The main objective of this paper is to give explicit sufficient conditions for existence of some or all of these classes of proper solutions of (1) defined on the given interval $[0, \infty)$. The qualitative behavior of solutions of equations of the form (1) has been the subject of numerous investigations; see, e.g., [1, 4, 6, 9, 10, 11, 12]. However, it seems that most of the existing literature has been concerned with equations which generalize the Emden–Fowler equations, and no systematic study of equations generalizing (2) or (3) has even been attempted. The present work was motivated by this observation. The main results will be developed in Section 2. In Section 3 we will show that the existence theory for (1) can be applied to prove existence theorems for the exterior Dirichlet problem for a class of second-order semilinear elliptic equations including $\Delta u = a(x)\,e^u$ as a special case, where $x \in R^n$ and Δ is the Laplace operator.

2. Main Results

We begin by giving a condition under which Eq. (1) has strongly decreasing solutions.

THEOREM 1. *Suppose that there exists a constant $c > 0$ such that*

$$\int_0^\infty f\left(t, -cP(t), -\frac{c}{p(t)}\right) dt < \infty. \tag{4}$$

Then, for any $b \in (c, \infty)$ and $\gamma \in R$, (1) has a strongly decreasing solution $y(t)$ satisfying

$$y(0) = \gamma \qquad \text{and} \qquad \lim_{t\to\infty} p(t)\,y'(t) = -b. \tag{5}$$

Proof. From (4) and (A2), we have

$$\int_0^\infty f\left(t, \gamma - bP(t), -\frac{b}{p(t)}\right) dt < \infty.$$

Let $C^1[0, \infty)$ denote the Fréchet space of all continuously differentiable functions on $[0, \infty)$ with the usual metric topology, and Y be the set of all $y \in C^1[0, \infty)$ that satisfy the following inequalities:

$$\gamma - bP(t) - \int_0^t \frac{1}{p(s)} \int_s^\infty f\left(r, \gamma - bP(r), -\frac{b}{p(r)}\right) dr\, ds \leqslant y(t) \leqslant \gamma - bP(t),$$

$$-b - \int_t^\infty f\left(s, \gamma - bP(s), -\frac{b}{p(s)}\right) ds \leqslant p(t)\, y'(t) \leqslant -b, \qquad t \geqslant 0.$$

Clearly, Y is a nonempty closed convex subset of $C^1[0, \infty)$. Define the operator $\mathscr{F}\colon Y \to C^1[0, \infty)$ by

$$\mathscr{F} y(t) = \gamma - bP(t) - \int_0^t \frac{1}{p(s)} \int_s^\infty f(r, y(r), y'(r))\, dr\, ds, \qquad t \geqslant 0.$$

It is easy to verify that $\mathscr{F} Y \subset Y$, $\mathscr{F}$ is continuous and $\overline{\mathscr{F} Y}$ is compact. So, the Schauder–Tychonoff fixed point theorem implies that $\mathscr{F}$ has a fixed point y in Y. This fixed point $y(t)$ is a strongly decreasing solution of (1) satisfying (5). This completes the proof.

Remark 1. If (1) has a strongly decreasing solution, then (4) holds for some $c > 0$.

We now prove a simple lemma which will be useful in the following discussions.

LEMMA 1. *Together with* (1) *we consider the equation*

$$(p(t)\, z')' = g(t, z, z'), \qquad t \geqslant 0, \tag{6}$$

where $p(t)$ is as in (1), *$g\colon [0, \infty) \times R \times R \to (0, \infty)$ is continuous and nondecreasing in the last two variables, and*

$$f(t, u, v) \geqslant g(t, u, v), \qquad (t, u, v) \in [0, \infty) \times R \times R.$$

Let $y(t)$ and $z(t)$ be solutions of (1) *and* (6), *respectively, satisfying $z(a) \leqslant y(a)$ and $z'(a) < y'(a)$. If $y(t)$ is defined on $[a, b)$, then $z(t)$ exists on $[a, b)$ and satisfies $z(t) < y(t)$ and $z'(t) < y'(t)$ for $t \in (a, b)$.*

Proof. First we assert that $z(t) < y(t)$ in (a, b). If this inequality does not hold, then there exists $t_0 \in (a, b)$ such that $z(t_0) = y(t_0)$, and $z(t) < y(t)$

for $t \in (a, t_0)$. Therefore $z'(t_0) \geqslant y'(t_0)$, and by the assumption, there exists $t_1 \in [a, t_0]$ such that $z'(t_1) = y'(t_1)$ and $z'(t) < y'(t)$ for $t \in [a, t_1)$. From (1) and (6) we then obtain

$$\begin{aligned} 0 &= p(t_1)[y'(t_1) - z'(t_1)] \\ &= p(a)[y'(a) - z'(a)] + \int_a^{t_1} [f(s, y(s), y'(s)) - g(s, z(s), z'(s))]\, ds \\ &= p(a)[y'(a) - z'(a)] + \int_a^{t_1} [f(s, y(s), y'(s)) - g(s, y(s), y'(s))]\, ds \\ &\quad + \int_a^{t_1} [g(s, y(s), y'(s)) - g(s, z(s), z'(s))]\, ds > 0, \end{aligned}$$

which is a contradiction. It follows that $z(t) < y(t)$ in (a, b). If we suppose that the inequality $z'(t) < y'(t)$ does not hold in (a, b), then, by considering the first point where $z'(t) = y'(t)$ and using the same argument as above, we are led to a contradiction. This completes the proof.

A sufficient condition for the existence of weakly decreasing solutions of (1) is given in the following theorem.

THEOREM 2. *Suppose that* (4) *holds for all* $c > 0$. *Then for any* $\gamma \in R$, (1) *has a unique weakly decreasing solution* $y(t)$ *satisfying* $y(0) = \gamma$.

Proof. We fix $\gamma \in R$. Let $y_\alpha(t)$ denote the solution of (1) satisfying $y(0) = \gamma$ and $p(0)\, y'(0) = \alpha$. We define the set $A \subset R$ by

$$A = \{\alpha \in R \colon y_\alpha(t) \text{ is a strongly decreasing solution}\},$$

which is nonempty by Theorem 1. Now we show that A is an open set which is bounded above. Let $\alpha \in A$. If $\beta < \alpha$, then by Lemma 1 $(g \equiv f)$, $y_\beta(t)$ is a strongly decreasing solution, that is, $\beta \in A$. Suppose that $\beta > \alpha$. Since $\alpha \in A$, there exists an $l > 0$ such that $\lim_{t \to \infty} p(t)\, y'_\alpha(t) = -l$. We choose $t_1 > 0$ large enough so that

$$\int_{t_1}^{\infty} f\left(t, \gamma - \frac{l}{2} P(t), -\frac{l}{2p(t)}\right) dt < \frac{l}{2}. \tag{7}$$

By the continuous dependence on initial conditions, for all $\beta > \alpha$ sufficiently close to α, $y_\beta(t)$ exist on $[0, t_1]$ and satisfy $p(t)\, y'_\beta(t) < -l$ for $t \in [0, t_1]$. It can be shown that for such a $\beta > \alpha$, $y_\beta(t)$ can be extended to $[0, \infty)$, and satisfies

$$p(t)\, y'_\beta(t) < -\frac{l}{2} \qquad \text{for} \quad t \geqslant 0. \tag{8}$$

In fact, if (8) fails, then there exists $t_2 > t_1$ such that

$$p(t_2)\, y'_\beta(t_2) = -\frac{l}{2} \qquad \text{and} \qquad p(t)\, y'_\beta(t) < -\frac{l}{2} \text{ for } t \in [0, t_2). \tag{9}$$

Integrating (1) and using (7), (8) and (9), we have

$$\begin{aligned}
-\frac{l}{2} &= p(t_2)\, y'_\beta(t_2) = p(t_1)\, y'_\beta(t_1) + \int_{t_1}^{t_2} f(s, y_\beta(s), y'_\beta(s))\, ds \\
&\leqslant -l + \int_{t_1}^{t_2} f\left(s, \gamma - \frac{l}{2} P(s), -\frac{l}{2p(s)}\right) ds \\
&\leqslant -l + \int_{t_1}^{\infty} f\left(s, \gamma - \frac{l}{2} P(s), -\frac{l}{2p(s)}\right) ds \\
&< -l + \frac{l}{2} \\
&= -\frac{l}{2}.
\end{aligned}$$

This contradiction proves that (8) holds, and this implies that $\beta \in A$. Thus A is open. On the other hand, if $\alpha \geqslant 0$, then $\alpha \notin A$, so that A is bounded from above. We put $\alpha^* = \sup A$. It is obvious that $\alpha^* \notin A$ and $\alpha^* \leqslant 0$.

We consider the solution $y_{\alpha^*}(t)$. By the continuous dependence on initial conditions, $y_{\alpha^*}(t)$ is not a singular solution, that is, $y_{\alpha^*}(t)$ exists on $[0, \infty)$ and satisfies $\lim_{t \to \infty} p(t)\, y'_{\alpha^*}(t) = \eta^* \geqslant 0$ (η^* may be ∞). The continuous dependence on initial conditions precludes the possibility that η^* is positive, and so we must have $\eta^* = 0$. This means that $y_{\alpha^*}(t)$ is a weakly decreasing solution passing through $(0, \gamma)$.

To prove the uniqueness of the weakly decreasing solution passing through $(0, \gamma)$, let $y_1(t)$ and $y_2(t)$ be two weakly decreasing solutions of (1) such that $y_1(0) = y_2(0) = \gamma$ but $y'_1(0) < y'_2(0)$. Lemma 1 ($g \equiv f$) implies that $y_1(t) \leqslant y_2(t)$ and $y'_1(t) \leqslant y'_2(t)$ for $t \geqslant 0$. It follows from (1) that

$$[p(t)(y'_2(t) - y'_1(t))]' = f(t, y_2(t), y'_2(t)) - f(t, y_1(t), y'_1(t)) \geqslant 0$$

for $t \geqslant 0$. So,

$$p(t)\, y'_2(t) - p(t)\, y'_1(t) \geqslant p(0)[y'_2(0) - y'_1(0)] > 0 \qquad \text{for} \quad t \geqslant 0.$$

Since the left-hand side of this inequality tends to 0 as $t \to \infty$, we have a contradiction. This completes the proof.

The following theorem gives a useful information about the asymptotic behavior of weakly decreasing solutions of (1).

THEOREM 3. *All weakly decreasing solutions of* (1), *if any, are either simultaneously bounded or simultaneously unbounded.*

Proof. Let $y_1(t)$ and $y_2(t)$ be weakly decreasing solutions satisfying $y_1(0)=\gamma_1$ and $y_2(0)=\gamma_2$ with $\gamma_1<\gamma_2$. It suffices to prove that the difference $y_2(t)-y_1(t)$ is a positive nonincreasing function on $[0, \infty)$. First we show that $y_2(t)>y_1(t)$ for $t\geqslant 0$. Otherwise, there exists $t_0>0$ such that $y_1(t_0)=y_2(t_0)$ and $y_1'(t_0)>y_2'(t_0)$. We choose $t_1>t_0$ sufficiently close to t_0 such that $y_1(t_1)>y_2(t_1)$ and $y_1'(t_1)>y_2'(t_1)$, and fix it. By the continuous dependence on initial data, a solution $\tilde{y}(t)$ of (1) with $\tilde{y}(0)=\gamma_2$ satisfies

$$y_2(t_1)<\tilde{y}(t_1)<y_1(t_1) \qquad \text{and} \qquad y_2'(t_1)<\tilde{y}'(t_1)<y_1'(t_1)$$

provided $\tilde{y}'(0)-y_2'(0)>0$ is sufficiently small. By Lemma 1, $\tilde{y}(t)$ exists on $[0, \infty)$ and satisfies $y_2'(t)<\tilde{y}'(t)<y_1'(t)$ for $t\geqslant t_1$, i.e.,

$$p(t)\,y_2'(t)<p(t)\,\tilde{y}'(t)<p(t)\,y_1'(t) \qquad \text{for} \quad t\geqslant t_1.$$

This fact means that $\tilde{y}(t)$ is a weakly decreasing solution passing through $(0, \gamma_2)$, which contradicts the uniqueness of the weakly decreasing solution passing through $(0, \gamma_2)$. Thus we obtain $y_2(t)>y_1(t)$ for $t\geqslant 0$. Next, if there exists $t_2\geqslant 0$ such that $y_1'(t_2)<y_2'(t_2)$, then the same argument as above leads us to the conclusion that there is a weakly decreasing solution different from $y_2(t)$ passing through $(0, \gamma_2)$. This again is a contradiction, and so we have $y_2'(t)\leqslant y_1'(t)$ for $t\geqslant 0$. It follows that $y_2(t)-y_1(t)$ is a positive nonincreasing function for $t\geqslant 0$, and the proof is complete.

Consider the case that the right-hand side of Eq. (1) does not include the first derivative y':

$$(p(t)y')'=h(t, y), \qquad t\geqslant 0, \tag{10}$$

where $p(t)$ satisfies condition (A1) and $h\colon [0, \infty)\times R\to(0, \infty)$ is continuous, nondecreasing in y. Theorem 2 shows that if

$$\int_0^\infty h(t, -cP(t))\,dt<\infty \qquad \text{for all} \quad c>0, \tag{11}$$

then for any $\gamma\in R$ there exists a unique weakly decreasing solution $y(t)$ of (10) satisfying $y(0)=\gamma$. In this case we can determine whether or not all the weakly decreasing solutions of (10) are bounded.

THEOREM 4. *Under the condition* (11), *all weakly decreasing solutions of* (10) *are bounded if and only if*

$$\int_0^\infty P(t)\,h(t, c)\,dt<\infty \qquad \textit{for some} \quad c\in R.$$

Proof. Since the "only if" part is easily proved, we only sketch the proof of the "if" part. Choose $k \geqslant 0$ large enough so that

$$\int_0^\infty \frac{1}{p(s)} \int_s^\infty h(r, -k)\, dr\, ds \leqslant k.$$

Consider the set

$$Y = \{ y \in C[0, \infty): -2k \leqslant y(t) \leqslant -k \text{ for } t \geqslant 0 \},$$

and define the operator $\mathscr{F}\colon Y \to C[0, \infty)$ by

$$\mathscr{F} y(t) = -2k + \int_t^\infty \frac{1}{p(s)} \int_s^\infty h(r, y(t))\, dr\, ds, \qquad t \geqslant 0.$$

From the Schauder–Tychonoff fixed point theorem $\mathscr{F}$ has a fixed point y in Y. Clearly this $y = y(t)$ is a bounded weakly decreasing solution on $[0, \infty)$ of equation (10), and so according to Theorem 3, all weakly decreasing solutions of (10) must be bounded.

Remark 2. Theorems 3 and 4 imply that under the condition (11), all weakly decreasing solutions of (10) are unbounded if and only if

$$\int_0^\infty P(t)\, h(t, c)\, dt = \infty \qquad \text{for all} \quad c \in R.$$

EXAMPLE 1. Consider equation (2). By Theorem 2, if

$$\int_0^\infty a(t)\, e^{-ct}\, dt < \infty \qquad \text{for all} \quad c > 0,$$

then for any $\gamma \in R$, (2) has a unique weakly decreasing solution passing through $(0, \gamma)$. If in addition

$$\int_0^\infty ta(t)\, dt < \infty \qquad \left(\text{or} \int_0^\infty ta(t)\, dt = \infty\right),$$

then by Theorem 4 all weakly decreasing solutions of (2) are bounded (or unbounded).

We now obtain conditions guaranteeing the existence of singular solutions of (1). The derivation is based on the techniques used by Kiguradze and Kvinikadze [5, Theorem 1.1].

THEOREM 5. *Suppose that there exists a positive continuous function $f_*(t, u, v)$ on $[0, \infty) \times R \times R$ which is nonincreasing in t and nondecreasing in u and v, and satisfies*

$$f(t, u, v) \geqslant f_*(t, u, v) \qquad \text{on } [0, \infty) \times R \times R.$$

Moreover suppose that $p(t)\,P(t)$ is nondecreasing. We define

$$F_\gamma(t, u) = \int_\gamma^u f_*\left(t, s, \frac{s-\gamma}{p(t)\,P(t)}\right) ds \qquad \text{for} \quad \gamma \in R,\ t > 0,\ u > \gamma.$$

If

$$\int^\infty (F_\gamma(t, u))^{-1/2}\, du < \infty \qquad \text{for any } t > 0,$$

then for every $t_0 \geqslant 0$, there exists a singular solution $y(t)$ of (1) satisfying $y(t_0) = \gamma$.

Proof. We fix $t_1 > t_0 \geqslant 0$. Let m and M be positive constants such that $m \leqslant p(t) \leqslant M$ for $t \in [t_0, t_1]$. Choose $\delta = \delta(\gamma, t_0) > 0$ large enough so that

$$M \int_\gamma^\infty (2mF_\gamma(t_1, u) + \delta^2)^{-1/2}\, du < t_1 - t_0. \tag{12}$$

Now we show that the solution $y(t)$ of (1) satisfying the initial conditions $y(t_0) = \gamma$ and $p(t_0)\, y'(t_0) \geqslant \delta$ cannot exist on $[t_0, t_1]$. Suppose the contrary. From (1) and the monotonicity of $p(t)\, y'(t)$, we see that

$$((p(t)\, y'(t))^2)' \geqslant 2p(t)\, y'(t) f_*(t, y(t), y'(t)), \qquad t \in [t_0, t_1]. \tag{13}$$

On the other hand, we have

$$\begin{aligned} y(t) &= \int_{t_0}^t y'(s)\, ds + \gamma = \int_{t_0}^t \frac{p(s)\, y'(s)}{p(s)}\, ds + \gamma \\ &\leqslant p(t)\, y'(t) \int_{t_0}^t \frac{ds}{p(s)} + \gamma \leqslant p(t) P(t)\, y'(t) + \gamma, \qquad t \in [t_0, t_1], \end{aligned}$$

that is,

$$y'(t) \geqslant \frac{y(t) - \gamma}{p(t)\, P(t)}, \qquad t \in [t_0, t_1]. \tag{14}$$

Integrating (13) from t_0 to $t \in [t_0, t_1]$ and using (14) and the monotonicity condition imposed on f_*, we obtain

$$(p(t)\,y'(t))^2 \geqslant 2m \int_{t_0}^{t} y'(s) f_*\left(s,\, y(s),\, \frac{y(s)-\gamma}{p(s)\,P(s)}\right) ds + \delta^2$$
$$\geqslant 2m \int_{t_0}^{t} y'(s) f_*\left(t,\, y(s),\, \frac{y(s)-\gamma}{p(t)\,P(t)}\right) ds + \delta^2$$

for $t \in [t_0, t_1]$, which is equivalent to

$$My'(t) \geqslant \left(2m \int_{\gamma}^{y(t)} f_*\left(t,\, s,\, \frac{s-\gamma}{p(t)\,P(t)}\right) ds + \delta^2\right)^{1/2}, \qquad t \in [t_0, t_1].$$

In view of the monotonicity of f_*, this implies

$$My'(t)(2mF_\gamma(t_1, y(t)) + \delta^2)^{-1/2} \geqslant 1, \qquad t \in [t_0, t_1].$$

Integrating from t_0 to t_1, and using (12) we obtain

$$t_1 - t_0 > M \int_{\gamma}^{\infty} (2mF_\gamma(t_1, u) + \delta^2)^{-1/2}\, du$$
$$\geqslant M \int_{\gamma}^{y(t_1)} (2mF_\gamma(t_1, u) + \delta^2)^{-1/2}\, du \geqslant t_1 - t_0,$$

a contradiction. Thus this solution $y(t)$ must be singular.

Remark 3. Note that in case the right-hand side of Eq. (1) does not depend on y', the monotonicity condition on $p(t)\,P(t)$ is unnecessary.

Remark 4. In the case $\gamma \geqslant 0$ we can choose $\delta > 0$ appearing in the proof of Theorem 5 to be independent of γ. This is a consequence of the following computation:

$$\int_{\gamma}^{\infty} (2mF_\gamma(t_1, u) + \delta^2)^{-1/2}\, du$$
$$= \int_0^{\infty} \left(2m \int_0^{v} f_*\left(t_1,\, s+\gamma,\, \frac{s}{p(t_1)\,P(t_1)}\right) ds + \delta^2\right)^{-1/2} dv$$
$$\leqslant \int_0^{\infty} \left(2m \int_0^{v} f_*\left(t_1,\, s,\, \frac{s}{p(t_1)\,P(t_1)}\right) ds + \delta^2\right)^{-1/2} dv$$
$$= \int_0^{\infty} (2mF_0(t_1, u) + \delta^2)^{-1/2}\, du.$$

We now turn to the problem of finding conditions for Eq. (1) to have weakly and strongly increasing solutions.

THEOREM 6. *Suppose that there exists a constant $c>0$ such that*

$$\int_0^\infty f\left(t, cP(t), \frac{c}{p(t)}\right) dt < \infty. \tag{15}$$

Then for any $b \in (0, c)$ and any $\gamma \in R$, Eq. (1) has a weakly increasing solution $y(t)$ satisfying

$$y(0) = \gamma \qquad \text{and} \qquad \lim_{t \to \infty} p(t)\, y'(t) = b. \tag{16}$$

Proof. The proof is virtually the same as that of Theorem 1. First note that

$$\int_0^\infty f\left(t, \gamma + bP(t), \frac{b}{p(t)}\right) dt < \infty.$$

Let Y denote the set of all $y \in C^1[0, \infty)$ satisfying the inequalities:

$$\gamma + bP(t) - \int_0^t \frac{1}{p(s)} \int_s^\infty f\left(r, \gamma + bP(r), \frac{b}{p(r)}\right) dr\, ds \leqslant y(t) \leqslant \gamma + bP(t),$$

$$b - \int_t^\infty f\left(s, \gamma + bP(s), \frac{b}{p(s)}\right) ds \leqslant p(t)\, y'(t) \leqslant b, \qquad t \geqslant 0.$$

Then the operator $\mathscr{F}\colon Y \to C^1[0, \infty)$ defined by

$$\mathscr{F}y(t) = \gamma + bP(t) - \int_0^t \frac{1}{p(s)} \int_s^\infty f(r, y(r), y'(r))\, dr\, ds, \qquad t \geqslant 0,$$

has a fixed point y in Y, which gives the desired weakly increasing solution of (1) satisfying (16).

Remark 5. If (1) has a weakly increasing solution, then (15) holds for some $c>0$.

The following is an analogue of a theorem of Kiguradze and Kvinikadze [5, Theorem 2.1].

THEOREM 7. *Suppose that the assumptions of Theorem 5 are satisfied for any $\gamma \in R$. If (15) holds for all $c>0$, then for any $\gamma \in R$, (1) has a strongly increasing solution $y(t)$ satisfying $y(0) = \gamma$.*

Proof. Let $\gamma \in R$ be fixed and let $y_\alpha(t)$ be the solution of (1) satisfying $y(0)=\gamma$ and $p(0)\, y'(0)=\alpha$. We define the sets $A, B \subset R$ by

$$A=\{\alpha \in R\colon y_\alpha(t) \text{ is a weakly increasing solution}\},$$

$$B=\{\alpha \in R\colon y_\alpha(t) \text{ is a singular solution}\}.$$

By Theorems 5 and 6, we see that $B \neq \varnothing$ and $A \neq \varnothing$. Lemma 1 implies that $\alpha \leqslant \beta$ for any $\alpha \in A$ and $\beta \in B$. Now we show that A and B are disjoint open subsets of R. Suppose that $\alpha \in A$. By the continuous dependence on initial data and Lemma 1, if $\beta<\alpha$ is sufficiently close to α, then $\beta \in A$. From the definition of A there exists $l>0$ such that $\lim_{t\to\infty} p(t)\, y'_\alpha(t)=l$. Choose $t_1>0$ large enough so that

$$\int_{t_1}^{\infty} f\left(t, \gamma+2lP(t), \frac{l}{p(t)}\right) dt<l.$$

From the continuous dependence on initial data, it follows that for all $\beta>\alpha$ sufficiently close to α, $y_\beta(t)$ exist on $[0, t_1]$ and satisfy $p(t)\, y'_\alpha(t)<l$ for $t \in [0, t_1]$. In the same manner as in the proof of Theorem 2, we can show that for such a $\beta>\alpha$, $y_\beta(t)$ exists on $[0, \infty)$ and satisfies $p(t)\, y'_\beta(t)<2l$ for $t \geqslant 0$. This shows that $\beta \in A$ provided $\beta>\alpha$ is sufficiently close to α, implying that A is open.

Next suppose that $\alpha \in B$. If $\beta>\alpha$, then $\beta \in B$ by Lemma 1. Let $\beta<\alpha$ and let $T>0$ be the point where $y_\alpha(t)$ fails to exist. According to Theorem 5 and Remark 4, there is a constant $\delta=\delta(T)>0$ such that all solutions $y(t)$ of (1) satisfying $y(T) \geqslant 0$ and $p(T)\, y'(T) \geqslant \delta$ are singular. Take $\varepsilon>0$ so that $y_\alpha(t)>0$ and $p(t)\, y'_\alpha(t) \geqslant \delta$ on $[T-\varepsilon, T)$. By the continuous dependence on initial data, if $\beta<\alpha$ is sufficiently close to α, then $y_\beta(t)$ exists on $[0, T-\varepsilon]$ and satisfies $y_\beta(T-\varepsilon)>0$ and $p(T-\varepsilon)\, y'_\beta(T-\varepsilon)>\delta$. Such a $y_\beta(t)$ is singular, because if $y_\beta(t)$ exists on $[0, T]$, it satisfies $y_\beta(T)>0$ and $p(T)\, y'_\beta(T)>\delta$. Thus B is also open.

Finally we put $\alpha^*=\sup A$ and $\beta_*=\inf B$. It is easily seen that $\alpha^* \notin A$, $\beta_* \notin B$, and $\alpha^* \leqslant \beta_*$. Then, for any $\alpha \in [\alpha^*, \beta_*]$ (which may be reduced to one point), $y_\alpha(t)$ is a strongly increasing solution of (1) satisfying $y(0)=\gamma$. This finishes the proof.

EXAMPLE 2. We consider (2) again. If

$$\int_0^{\infty} a(t)\, e^{ct}\, dt<\infty \qquad \text{for all} \quad c>0$$

(for example $a(t)=\exp(-kt^q)$, $k>0$, $q>1$), then for any $\gamma \in R$, (2) has all kinds of proper solutions satisfying $y(0)=\gamma$ as well as singular solutions.

3. Applications

In this section we show that the above results for the ordinary differential equation (1) can be applied to solve the elliptic exterior Dirichlet problem

$$\Delta u = a(x)\,\phi(u), \qquad x \in \Omega, \tag{17}$$

$$u = \gamma(x), \qquad x \in \partial\Omega, \tag{18}$$

where Ω is an exterior domain in R^n ($n \geqslant 2$) with boundary $\partial\Omega$ and Δ is the n-dimensional Laplacian. The following hypotheses are assumed to hold:

(B1) $\partial\Omega$ is of class $C^{2+\theta}$, $\theta \in (0, 1)$;

(B2) $a\colon R^n \to (0, \infty)$ is locally Hölder continuous with exponent θ;

(B3) $\phi\colon R \to (0, \infty)$ is locally Lipschitz continuous and non-decreasing;

(B4) $\gamma\colon \partial\Omega \to R$ is of class $C^{2+\theta}(\partial\Omega)$.

An important special case of (17) satisfying (B1)–(B3) is

$$\Delta u = a(x)\, e^u, \qquad x \in \Omega.$$

A powerful tool for solving the exterior Dirichlet problem for second-order elliptic equations is the supersolution–subsolution method which, when specialized to our structure, reads:

Lemma 2. *If there exist functions $v, w \in C^{2+\theta}_{\mathrm{loc}}(\bar{\Omega})$ such that*

$$\Delta v(x) \leqslant a(x)\,\phi(v(x)) \quad \text{in } \Omega, \qquad v(x) \geqslant \gamma(x) \text{ on } \partial\Omega, \tag{19}$$

$$\Delta w(x) \geqslant a(x)\,\phi(w(x)) \quad \text{in } \Omega, \qquad w(x) \leqslant \gamma(x) \text{ on } \partial\Omega, \tag{20}$$

$$w(x) \leqslant v(x) \qquad \text{in } \bar{\Omega}, \tag{21}$$

then the problem (17)–(18) *has at least one solution $u \in C^{2+\theta}_{\mathrm{loc}}(\bar{\Omega})$ such that $w(x) \leqslant u(x) \leqslant v(x)$ in $\bar{\Omega}$.*

A proof of Lemma 2 can be found in [8, Theorem 3.3]. A function v (resp. w) satisfying (19) (resp. (20)) is called a supersolution (resp. subsolution) of the problem (17)–(18). Lemma 2 asserts that the problem (17)–(18) is solved if a supersolution v and a subsolution w satisfying (21) are shown to exist. Our purpose here is to construct suitable spherically symmetric supersolutions and subsolutions with the aid of the existence theory for (1), thereby proving the existence of solutions of the exterior problem under study with various asymptotic behaviors at infinity. For related existence results the reader is referred to [2, 3, 6, 7].

In what follows the next lemma is needed.

LEMMA 3. *Consider the ordinary differential equations* (1) *and* (6) *under the same conditions as in Lemma* 1. *Let* $y(t)$ *and* $z(t)$ *be solutions of* (1) *and* (6), *respectively, satisfying*

$$y(0) \leqslant z(0),$$
$$\lim_{t \to \infty} p(t)\, y'(t) = \lim_{t \to \infty} p(t)\, z'(t) = l \in R.$$

Then $y(t) \leqslant z(t)$ *for* $t \geqslant 0$.

Proof. Suppose that $y(t) > z(t)$ for some $t > 0$. Then there exists $t_1 > 0$ such that $y(t_1) > z(t_1)$ and $y'(t_1) > z'(t_1)$. By Lemma 1, $y(t) > z(t)$ and $y'(t) > z'(t)$ for $t \geqslant t_1$, and so from (1) and (6) we have

$$\begin{aligned}&(p(t)\, y'(t) - p(t)\, z'(t))' \\ &\quad = f(t, y(t), y'(t)) - g(t, z(t), z'(t)) > 0 \qquad \text{for} \quad t \geqslant t_1,\end{aligned}$$

from which it follows that

$$p(t)\, y'(t) - p(t)\, z'(t) \geqslant p(t_1)[\, y'(t_1) - z'(t_1)] > 0$$

for $t \geqslant t_1$. This is a contradiction since the left-hand side of the above inequality must tend to 0 as $t \to \infty$.

Remark 6. Lemma 3 implies the uniqueness of the solution of (1) with prescribed initial and terminal conditions:

$$y(0) = \gamma \qquad \text{and} \qquad \lim_{t \to \infty} p(t)\, y'(t) = l \in R.$$

Let us now study the exterior Dirichlet problem (17)–(18). Without loss of generality we may assume that the origin is contained in $\bar{\Omega}^c$. We employ the following notation: T denotes the radius of the largest sphere centered at the origin and contained in Ω^c, and T' denotes the radius of the smallest sphere centered at the origin and containing Ω^c;

$$G_T = \{x \in R^n\colon |x| > T\}, \qquad \partial G_T = \{x \in R^n\colon |x| = T\};$$
$$a^*(t) = \max_{|x| = t} a(x), \qquad a_*(t) = \min_{|x| = t} a(x), \quad t \geqslant T;$$
$$\gamma^* = \max_{\partial\Omega} \gamma(x), \qquad \gamma_* = \min_{\partial\Omega} \gamma(x).$$

Let $y, z \in C^{2+\theta}_{\text{loc}}[T, \infty)$ be solutions of the ordinary differential equations

$$(t^{n-1}y')' = t^{n-1}a_*(t)\,\phi(y), \qquad t \geqslant T, \tag{22}$$

$$(t^{n-1}z')' = t^{n-1}a^*(t)\,\phi(z), \qquad t \geqslant T, \tag{23}$$

respectively. Then the functions $v(x) = y(|x|)$ and $w(x) = z(|x|)$ satisfy $\Delta v \leqslant a(x)\,\phi(v)$ and $\Delta w \geqslant a(x)\,\phi(w)$ in G_T, so that if $w \leqslant \gamma(x) \leqslant v$ on $\partial\Omega$, then $v(x)$ and $w(x)$ are, respectively, a supersolution and a subsolution of the problem (17)–(18). Our task below is to construct the desired super- and subsolution with specified asymptotic behavior by using an appropriate theorem in Section 2 specialized to (22) and (23).

We begin with the two-dimensional case.

THEOREM 8. $(n = 2)$ (i) *Suppose that there is a constant $c > 0$ such that*

$$\int_T^\infty ta^*(t)\,\phi(-c\log t)\,dt < \infty. \tag{24}$$

Then, for any $b \in (c, \infty)$, the problem (17)–(18) *has a solution $u(x)$ satisfying*

$$\lim_{|x| \to \infty} \frac{u(x)}{\log|x|} = -b.$$

(ii) *Suppose that* (24) *holds for all $c > 0$. If in addition*

$$\int_T^\infty ta^*(t)\log t\,dt < \infty, \tag{25}$$

then the problem (17)–(18) *has a bounded solution. If* (25) *is replaced by*

$$\int_T^\infty ta_*(t)\log t\,dt = \infty, \tag{26}$$

then the problem (17)–(18) *has a solution $u(x)$ satisfying*

$$\lim_{|x| \to \infty} u(x) = -\infty \qquad \textit{and} \qquad \lim_{|x| \to \infty} \frac{u(x)}{\log|x|} = 0.$$

(iii) *Suppose that there is a constant $c > 0$ such that*

$$\int_T^\infty ta^*(t)\,\phi(c\log t)\,dt < \infty.$$

Then, for any $b \in (0, c)$, the problem (17)–(18) *has a solution $u(x)$ satisfying*

$$\lim_{|x| \to \infty} \frac{u(x)}{\log|x|} = b.$$

Proof. (i) Let $y(t)$ and $z(t)$ be the solutions of (22) and (23) satisfying

$$y(T) = \gamma^*, \qquad \lim_{t \to \infty} \frac{y(t)}{\log t} = -b$$

and

$$z(T) = \gamma_*, \qquad \lim_{t \to \infty} \frac{z(t)}{\log t} = -b,$$

respectively. The existence of $y(t)$ and $z(t)$ is guaranteed by Theorem 1 specialized to (22) and (23). Applying Lemma 3, we see that $y(t) \geqslant z(t)$ for $t \geqslant T$. We now choose a constant $C > 0$ large enough so that

$$y(t) + C \geqslant \gamma^* \qquad \text{and} \qquad z(t) - C \leqslant \gamma_* \quad \text{for } t \in [T, T'], \tag{27}$$

and put

$$\tilde{y}(t) = y(t) + C \qquad \text{and} \qquad \tilde{z}(t) = z(t) - C \quad \text{for } t \geqslant T.$$

In view of (27) we have $\tilde{y}(|x|) \geqslant \gamma^* \geqslant \gamma(x) \geqslant \gamma_* \geqslant \tilde{z}(|x|)$ on $\partial\Omega$. Since

$$(t\tilde{y}'(t))' = ta_*(t)\,\phi(\tilde{y}(t) - C) \leqslant ta_*(t)\,\phi(\tilde{y}(t)),$$
$$(t\tilde{z}'(t))' = ta^*(t)\,\phi(\tilde{z}(t) + C) \geqslant ta^*(t)\,\phi(\tilde{z}(t))$$

for $t \geqslant T$, the functions $v(x) = \tilde{y}(|x|)$ and $w(x) = \tilde{z}(|x|)$ are, respectively, a supersolution and a subsolution of the problem (17)–(18). The conclusion of (i) now follows from Lemma 2.

(ii) From Theorem 2 applied to (22) and (23) it follows that (22) and (23) have solutions $y(t)$ and $z(t)$ such that

$$y(T) = \gamma^* \qquad \text{and} \qquad \lim_{t \to \infty} \frac{y(t)}{\log t} = 0$$

and

$$z(T) = \gamma_* \qquad \text{and} \qquad \lim_{t \to \infty} \frac{z(t)}{\log t} = 0,$$

respectively. Proceeding as in (i), we obtain a supersolution $v(x) = y(|x|) + C$ and a subsolution $w(x) = z(|x|) - C$ such that $w(x) \leqslant v(x)$ in $\bar{\Omega}$.

Thus Lemma 2 insures that the problem (17)–(18) has a solution $u(x)$ such that $w(x) \leqslant u(x) \leqslant v(x)$ in $\bar{\Omega}$. That condition (25) (resp. (26)) implies the boundedness (resp. unboundedness) of $u(x)$ is a consequence of Theorem 4.

(iii) Apply the same argument as in the proof of (i) by using the solutions $y(t)$ and $z(t)$ of (22) and (23) satisfying

$$y(T) = \gamma^*, \qquad \lim_{t \to \infty} \frac{y(t)}{\log t} = b,$$

and

$$z(T) = \gamma_*, \qquad \lim_{t \to \infty} \frac{z(t)}{\log t} = b,$$

respectively, which are guaranteed by Theorem 6. This completes the proof.

In the case of higher dimensions we have the following theorem.

THEOREM 9. $(n \geqslant 3)$ *Suppose that*

$$\int_T^\infty t a^*(t)\, dt < \infty. \tag{28}$$

Then, for any $b \in R$, *the problem* (17)–(18) *has a solution* $u(x)$ *such that* $\lim_{|x| \to \infty} u(x) = b$.

If in addition

$$\int_T^\infty t^{n-1} a^*(t)\, dt < \infty, \tag{29}$$

then the solution $u(x)$ *with* $b = 0$ *satisfies* $u(x) = O(|x|^{2-n})$ *as* $|x| \to \infty$. *If* (29) *is replaced by*

$$\int_T^\infty t^{n-1} a_*(t)\, dt = \infty,$$

then the solution $u(x)$ *with* $b = 0$ *satisfies* $\lim_{|x| \to \infty} |x|^{n-2} u(x) = -\infty$.

Proof. In the case $n \geqslant 3$ we rewrite (22) and (23) as follows:

$$(t^{3-n}\eta')' = t a_*(t)\, \phi(t^{2-n}\eta), \qquad t \geqslant T, \tag{30}$$

$$(t^{3-n}\zeta')' = t a^*(t)\, \phi(t^{2-n}\zeta), \qquad t \geqslant T, \tag{31}$$

where $\eta = t^{n-2} y$ and $\zeta = t^{n-2} z$. Since (28) implies the hypotheses of Theorems 1, 2, and 6 for (30) and (31), we conclude from these theorems that, for any $b \in R$, equations (30) and (31) have solutions $\eta(t)$ and $\zeta(t)$ satisfying

$$\eta(T) = T^{n-2}\gamma^*, \qquad \lim_{t \to \infty} t^{2-n}\eta(t) = b,$$

and

$$\zeta(T) = T^{n-2}\gamma_*, \qquad \lim_{t \to \infty} t^{2-n}\zeta(t) = b,$$

respectively. By Lemma 3 we have $\eta(t) \geqslant \zeta(t)$ for $t \geqslant T$. Arguing exactly as in the proof of (i) of Theorem 8, we see that, for some constant $C > 0$, the functions $\tilde{\eta}(t) = \eta(t) + C$ and $\tilde{\zeta}(t) = \zeta(t) - C$ satisfy

$$\begin{aligned}
&\tilde{\eta}(t) \geqslant \tilde{\zeta}(t), && t \geqslant T,\\
&\tilde{\eta}(t) \geqslant t^{n-2}\gamma^* \geqslant t^{n-2}\gamma_* \geqslant \tilde{\zeta}(t), && t \in [T, T'],\\
&(t^{3-n}\tilde{\eta}'(t))' \leqslant t a_*(t)\, \phi(t^{2-n}\tilde{\eta}(t)), && t \geqslant T,\\
&(t^{3-n}\tilde{\zeta}'(t))' \geqslant t a^*(t)\, \phi(t^{2-n}\tilde{\zeta}(t)), && t \geqslant T.
\end{aligned}$$

Put $\tilde{y}(t) = t^{2-n}\tilde{\eta}(t)$, $\tilde{z}(t) = t^{2-n}\tilde{\zeta}(t)$ and define $v(x) = \tilde{y}(|x|)$ and $w(x) = \tilde{z}(|x|)$. Then, $v(x)$ and $w(x)$ are a supersolution and a subsolution of the problem (17)–(18) such that $w(x) \leqslant v(x)$ in $\bar{\Omega}$ and $\lim_{|x| \to \infty} v(x) = \lim_{|x| \to \infty} w(x) = b$, and hence we have the desired conclusion from Lemma 2. This proves the first statement of the theorem. The remaining statements can be proved easily with the help of Theorem 4. This finishes the proof.

A natural question arises: Is it possible to apply Theorem 7 to obtain conditions under which the problem (17)–(18) has a solution $u(x)$ with the property

$$\lim_{|x| \to \infty} \frac{u(x)}{\log |x|} = \infty \tag{32}$$

in the case $n = 2$, and

$$\lim_{|x| \to \infty} u(x) = \infty \tag{33}$$

in the case $n \geqslant 3$? A partial answer to this question is given in the following theorem.

THEOREM 10. *Consider the particular exterior Dirichlet problem*

$$\Delta u = a(x)\, e^u, \qquad x \in \Omega, \tag{34}$$

$$u = \gamma(x), \qquad x \in \partial\Omega, \tag{35}$$

where, in addition to (B2), $a(x)$ *satisfies a more restrictive condition*

$$\limsup_{t \to \infty} \frac{a^*(t)}{a_*(t)} < \infty. \tag{36}$$

(i) *Let* $n=2$. *If*

$$\int_T^\infty t^{1+c} a^*(t)\, dt < \infty \qquad \textit{for all} \quad c>0, \tag{37}$$

then the problem (34)–(35) *has a solution* $u(x)$ *satisfying* (32).

(ii) *Let* $n \geqslant 3$. *If* (28) *holds, then the problem* (34)–(35) *has a solution* $u(x)$ *satisfying* (33).

Proof. Because of (36), there is a constant $C \geqslant 1$ such that $a^*(t) \leqslant C a_*(t)$ for $t \geqslant T$. We may assume that C is large enough so that we require.

(i) Let $n=2$. In this case (22) becomes

$$(ty')' = t a_*(t)\, e^y, \qquad t \geqslant T. \tag{38}$$

By Theorem 7, condition (37) ensures that (38) has a solution $y(t)$ such that $y(T) = \gamma^*$ and $\lim_{t \to \infty} y(t)/\log t = \infty$. For some constant $C_0 > 0$, the function $\tilde{y}(t) = y(t) + C_0$ satisfies

$$\begin{aligned} &\tilde{y}(t) \geqslant \gamma^*, && t \in [T, T'], \\ &(t\tilde{y}'(t))' \leqslant t a_*(t)\, e^{\tilde{y}(t)}, && t \geqslant T. \end{aligned}$$

On the other hand there is a constant $C \geqslant 1$ such that

$$y(t) - \log C \leqslant \gamma_*, \qquad t \in [T, T'],$$

and

$$a^*(t) \leqslant C a_*(t), \qquad t \geqslant T.$$

Put $\tilde{z}(t) = y(t) - \log C$. We then see that $\tilde{z}(t)$ satisfies

$$\begin{aligned} &\tilde{z}(t) \leqslant \gamma_*, && t \in [T, T'], \\ &(t\tilde{z}'(t))' \geqslant t a^*(t)\, e^{\tilde{z}(t)}, && t \geqslant T. \end{aligned}$$

Thus, $v(x) = \tilde{y}(|x|)$ and $w(x) = \tilde{z}(|x|)$ are a supersolution and a subsolution, respectively, of the problem (34)–(35) satisfying $v(x) \geqslant w(x)$ on $\bar{\Omega}$. The conclusion follows from Lemma 2 immediately.

(ii) In the case $n \geqslant 3$, after the transformation $\eta = t^{n-2}y$, (22) becomes

$$(t^{3-n}\eta')' = ta_*(t)\exp(t^{2-n}\eta), \qquad t \geqslant T. \tag{39}$$

We can apply Theorem 7 to obtain a solution $\eta(t)$ of (39) such that $\eta(T) = T^{n-2}\gamma^*$ and $\lim_{t\to\infty} t^{2-n}\eta(t) = \infty$. Choose $C_0 > 0$ large enough so that $\eta(t) + C_0 \geqslant t^{n-2}\gamma^*$ for $t \in [T, T']$. Then the function $\tilde{y}(t) = t^{2-n}(\eta(t) + C_0)$ satisfies

$$\begin{aligned} &\tilde{y}(t) \geqslant \gamma^*, && t \in [T, T'], \\ &(t^{n-1}\tilde{y}'(t))' \leqslant t^{n-1}a_*(t)e^{\tilde{y}(t)}, && t \geqslant T. \end{aligned}$$

Next, choose $C \geqslant 1$ large enough so that $\eta(t) \leqslant t^{n-2}(\gamma_* + \log C)$ for $t \in [T, T']$ and $a^*(t) \leqslant Ca_*(t)$ for $t \geqslant T$. Then, the function $\tilde{z}(t) = t^{2-n}\eta(t) - \log C$ satisfies

$$\begin{aligned} &\tilde{z}(t) \leqslant \gamma_*, && t \in [T, T'], \\ &(t^{n-1}\tilde{z}'(t))' \geqslant t^{n-1}a^*(t)\, e^{\tilde{z}(t)}, && t \geqslant T. \end{aligned}$$

Thus the functions $v(x) = \tilde{y}(|x|)$ and $w(x) = \tilde{z}(|x|)$ are a supersolution and a subsolution of the problem (34)–(35) which ensure the existence of a solution $u(x)$ satisfying (33).

Remark 7. From Theorems 8 and 9 we see that under the hypotheses of Theorem 10 the problem (34)–(35) also has, for every $b \in R$, a solution $u(x)$ satisfying $\lim_{|x|\to\infty} u(x)/\log|x| = b$ for $n = 2$ and $\lim_{|x|\to\infty} u(x) = b$ for $n \geqslant 3$.

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